

Calculus (Tutorial # 1)

Real number system

1. Let A and B be two nonempty sets. Let $f : A \mapsto B$ be a function and E and F are subsets of A . Then prove that

(a) $f(E \cup F) = f(E) \cup f(F)$

(b) $f(E \cap F) \subseteq f(E) \cap f(F)$.

2. Let A and B be two nonempty sets and $f : A \mapsto B$ be a function. For $H \subseteq B$, we define the inverse image of H under f is the subset $f^{-1}(H)$ of A given by $f^{-1}(H) := \{x \in A : f(x) \in H\}$. Now let G and H are subsets of B then prove the following

(a) $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$

(b) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$.

3. Let $x > 0$ and $n \in \mathbb{N}$, then show that $x^{\frac{1}{n}} \leq \frac{x + n - 1}{n}$.

4. Let $a_i \in \mathbb{R}$ for $1 \leq i \leq n$ and $b_j \in \mathbb{R}$ for $1 \leq j \leq n$ then show that

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \leq \left(\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \right) \left(\sqrt{b_1^2 + b_2^2 + \cdots + b_n^2} \right).$$

5. What would you choose for m in $|\sqrt{3} - 1.732| < 10^{-m}$, and why?

6. Prove that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$, for all $n \in \mathbb{N}$ with $n > 1$.

7. Let $x_1 := 1$, $x_2 := 2$ and $x_{n+2} := \frac{1}{2}(x_{n+1} + x_n)$ for all $n \in \mathbb{N}$. Then show that $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$.

8. **Bernoulli's Inequality:** If $x > -1$, then $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

9. Recall from the class that for $a \in \mathbb{R}$ and $\epsilon > 0$ we defined the ϵ -neighborhood of a is the set $V_\epsilon(a) := \{x \in \mathbb{R} : |x - a| < \epsilon\}$. Now solve the following problems:

- (a) Let $\epsilon > 0$ and $\delta > 0$ and $a \in \mathbb{R}$. Show that $V_\epsilon(a) \cap V_\delta(a)$ and $V_\epsilon(a) \cup V_\delta(a)$ are γ -neighborhoods of a for some appropriate value of γ . Also find γ in each case.

- (b) Let $\epsilon_n = \frac{1}{n}$ for $n \in \mathbb{N}$ and $a \in \mathbb{R}$. Let $V_{\epsilon_n}(a)$ be neighborhoods of a for $n \in \mathbb{N}$.

Then is it true that $\bigcap_{n=1}^{\infty} V_{\epsilon_n}(a)$ is a neighborhood of a ? What about $\bigcup_{n=1}^k V_{\epsilon_n}(a)$?

- (c) Let $a, b \in \mathbb{R}$ with $a \neq b$. Then can you find ϵ -neighborhoods $V_\epsilon(a)$ of a and $V_\epsilon(b)$ of b such that $V_\epsilon(a) \cap V_\epsilon(b) = \phi$?

10. Prove that a non-empty subset A of \mathbb{R} is bounded if and only if there is a positive real number M such that $-M \leq a \leq M$ for all $a \in A$ (i.e. $|a| \leq M$).

11. If A is a non-empty subset of \mathbb{R} which is bounded above then show that $-A := \{-x : x \in A\}$ is bounded below and $\sup(A) = -\inf(-A)$.
12. Let A and B be two nonempty bounded above subsets of \mathbb{R} then prove that $A + B := \{a + b : a \in A \text{ and } b \in B\}$ is also bounded above subset of \mathbb{R} and $\sup(A + B) = \sup A + \sup B$. How about $A - B := \{a - b : a \in A \text{ and } b \in B\}$? Is it bounded above?
13. Let $\alpha \in \mathbb{R}$ be non-negative and $n \in \mathbb{N}$. Then there exists a unique non-negative $x \in \mathbb{R}$ such that $x^n = \alpha$.
14. (Nested Interval Theorem) Let $J_n := [a_n, b_n]$ be intervals in \mathbb{R} such that $J_{n+1} \subseteq J_n$ for all $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} J_n \neq \emptyset$.
15. Find the supremum (l.u.b.) and infimum (g.l.b) if they exist, of each of the following sets.

$$(a) \left\{ 1 - \frac{1}{n^2} : n \in \mathbb{N} \right\} \quad (b) \{x \in \mathbb{R} : x^2 - 5x + 6 < 0\} \quad (c) \{x + x^{-1} : x > 0\}$$

$$(d) \left\{ \frac{m+n}{mn} : m, n \in \mathbb{N} \right\} \quad (e) \{r \in \mathbb{Q} : r < a, \} \text{ for } a \in \mathbb{R} \text{ fixed}$$

$$(f) \bigcup_{n \in \mathbb{N}} [2n, 2n+1] \quad (g) \{n^{-1} : n \in \mathbb{N} \text{ and } n \text{ is prime}\} \quad (h) \left\{ \sin\left(\frac{n\pi}{3}\right) : n \in \mathbb{N} \right\}$$

$$(i) \bigcap_{n \in \mathbb{N}} \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right) \quad (j) \left\{ 1 - \frac{1}{3^n} : n \in \mathbb{N} \right\} \quad (k) \left\{ 1 - \frac{(-1)^n}{n} : n \in \mathbb{N} \right\}$$

$$(l) \{x \in \mathbb{R} : |x+1| + |x-2| = 7\}.$$